IN PRAISE OF FULL MEET CONTRACTION

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Abstract

Full meet contraction, that was devised by Carlos Alchourrón and David Makinson in the early 1980's, has often been overlooked since it is not in itself a plausible contraction operator. However, it is a highly useful building-block in the construction of composite contraction operators. In particular, all plausible contraction operators can be reconstructed so that the outcome of contracting a belief set K by a sentence *p* is defined as K ~ *f* (p), where ~ is full meet contraction and *f* a sentential selector, i.e. a function to and from sentences. This paper investigates the logic of full meet contraction. Seven properties of this operation are presented that contribute to making it useful as a building-block: (1) Full meet contraction is a purely logical operation. (2) It retains finite-basedness of the belief set. (3) It is the inclusion-maximal contraction that removes all sentences that can contribute to implying the input sentence. (4) It is the inclusion-maximal contraction that removes all non-tautologous consequences of the input sentence. (5) Almost all contractions can be reconstructed as full meet contraction. (6) Full meet contraction allows for recovery of the input sentence. (7) Full meet contraction provides a unified account of multiple and singleton contraction.

KEY WORDS: belief change - full meet contraction - specified meet contraction - sentential selector.

Resumen

La llamada full meet contraction, creada por Carlos Alchourrón y David Makinson en los comienzos de los '80, ha sido a menudo desestimada puesto que ella no constituye un operador de contracción plausible. De todas formas ella es altamente útil en la construcción de operadores de contracción compuestos. En particular, todos los operadores de contracción plausibles pueden ser reconstruidos de forma tal que la contracción de un conjunto de creencias K por una sentencia p sea definida como K ~ *f*(p), donde ~ es una full meet contraction y f un selector de sentencias, i.e., una función de sentencias a sentencias. En este trabajo investigamos la lógica de la full meet contraction. Se presentan siete propiedades de esta operación que contribuyen a hacerla útil como building-block: (1) Full meet contraction es una operación puramente lógica.(2) Ella retiene la finit- basednes del conjunto de creencias.(3) Ella es la máxima-inclusiva de las contracciones que elimina todas las sentencias que pueden contribuir a implicar la input-sentence. (4) Es la contracción maximal inclusiva que elimina todas las consecuencias no tautológicas de la input-sentence. (5) Casi todas las contracciones pueden ser reconstruidas como full-meet contractions. (6) Full meet contraction permite la recuperación de la input-sentence.(7) Full meet contracción da cuenta en forma unificada de las contracciones múltiples y singulares.

PALABRAS CLAVE: cambio de creencia - *full meet contraction* - *specified meet contraction* - selector sentencial.

1. Introduction

Full meet contraction (originally called "meet contraction") was among the first contraction operators to be proposed in the belief change literature. For any set A and sentence p, let A⊥p be the set of maximal subsets of A that do not imply p. The full meet contraction of A by p, written A ~ p, is defined as ∩(A ⊥ p, except in the limiting case when p is a tautology (and consequently A ⊥p is empty), in which case A ~ p = A.

Full meet contraction was introduced by Carlos Alchourrón and David Makinson in 1982. However, after introducing it they were quick to denounce it as implausible:

> "It turns out that the [full] meet functions give sets that are far too weak to serve our purposes, whether they be applied to theories or to their bases. ... When we apply [full] meet contraction to a theory, we find that, under broad and natural assumptions on the structure of Cn, the function adds nothing of interest to the underlying consequence relation, as we now show." (Alchourrón and Makinson, 1982, p. 18)

This was followed by the following formal result:

OBSERVATION 1. (Alchourrón and Makinson, 1982) Let K be a logically closed set and ~ the operator of full meet contraction for K Then for all sentences p ∈ K.:
K ~ p = K∩ Cn ({¬p})

PROOF. See Alchourrón and Makinson, 1982, pp. 18-19 or Hansson, 1999, pp. 125-126.

This is indeed a highly implausible property for an operation intended to represent the types of belief contraction that we perform in our daily lives.

In the classical AGM article from 1985, written jointly by Carlos Alchourrón, Peter Gärdenfors and David Makinson, this argument against full meet contraction was repeated.

> "In the search for suitable [contraction] functions with smaller values [than maxichoice contraction], it is tempting to try the operation A ~ x defined as ∩ (A ⊥ x) when A ⊥ x is nonempty, and as A itself in the limiting case when A ⊥ x is empty. But as shown in [our Obser-

vation 1], this set is in general far too *small*. In particular, when A is a theory with $x \in A$, then $A \sim x = A \cap Cn(\neg x)$. In other words, the only propositions left in $A \sim x$ when A is a theory containing x are those which are already consequences of ¬x considered alone." (Alchourrón, Gärdenfors and Makinson, 1985, p. 512)

However, the authors were willing to endorse the use of full meet contraction as a reference point:

"Nevertheless, the operation of [full] *meet contraction*, as we shall call ~ is very useful as a point of reference. It serves as a natural *lower bound* on any reasonable contraction operation: any contraction operación ÷ worthy of the name should surely have $A \sim x \subseteq A \div x$ for all A, x, and a function ÷ satisfying this condition for a given A will be called *bounded over A*."(Alchourrón, Gärdenfors, and Makinson 1985, p. 512)

Instead of full meet contraction, Alchourrón and coworkers advanced the operator of *partial meet contraction*. A partial meet contraction for A is based on a selection function γ such that for all sentences p, $\gamma(A\perp p)$ is asubset of $A \perp p$ unless $A \perp p$ is empty, in which case $\gamma(A\perp p) = \{A\}$. The contraction operator ÷ for A is a partial meet contraction if and only if there is a selection function γ for A such that for all p, $A \div p = \cap\, \gamma(A\perp p)$. If for all p , $\gamma(A\perp p)$ has exactly one element, then ÷ is a *maxichoice* contraction.

This contribution is devoted to arguing that full meet contraction has interesting properties that make it useful not only as a "point of reference" but also as a building-block for contraction operators (in much the same way as the simplistic operationof expansion is an indispensible building-block in the construction of more sophisticated revision operators). Although elements from full meet contraction are used in partial meet contraction, full meet contraction is not used as a building-block here since it is broken into two parts between which a selection function is inserted. I have recently (Hansson 2006a, 2006b) proposed an alternative operation, *specified meet contraction*. It is based on a sentential selector, a function from and to sentences. An operator ÷ is a specified meet contraction if and only if there is a sentential selector f such that it holds for all p that $A \div p = A \sim f(p)$. Here, full meet contraction is used, unbroken, as a building-block to be applied after the sentential selection has been performed.

After formal preliminaries are dealt with in Section 2, seven virtues of full meet contraction are introduced in Section 3, i.e. seven properties that contribute to its usefulness as a building-block in composite operators of change. Some concluding remarks are offered in Section 4.

2. Formal preliminaries

The belief-representing sentences form a language L. Sentences, i.e. elements of this language, are represented by lowercase letters (p,q,…) and sets of sentences by capital letters. The language contains the usual truth-functional connectives: negation (¬), conjunction (&), disjunction (∨), implication (→), and equivalence (↔).

To express the logic, a Tarskian consequence operator Cn will be used. Intuitively speaking, for any set A of sentences, Cn(A) is the set of logical consequences of A is a function from sets of sentences to sets of sentences. It satisfies the standard conditions: inclusion ($A \subseteq Cn(A)$), monotony (If $A \subseteq B$, then $Cn(A) \subseteq Cn(B)$) and iteration ($Cn(A) = Cn(Cn(A))$). Furthermore, Cn is assumed to be supraclassical (if p follows from A by classical truth-functional logic, then $p \in Cn(A)$), and to satisfy the deduction property ($q \in Cn(A \cup \{p\})$ if and only if $(p \rightarrow q) \in Cn(A)$).

A is a belief set if and only if A = Cn(A). A logically closed set A is *finite-based* if and only if there is some finite set A' such that A = Cn(A'). For any set A such that Cn(A) is finite-based, &A is a sentence such that $Cn(A) = Cn(\{\&A\})$. K denotes a belief set. $Cn(\emptyset)$ is the set of tautologies. $X \vdash p$ is an alternative notation for $p \in Cn(X)$ and $\vdash p$ for $p \in Cn(\emptyset)$.

For any sets A and X, the remainder set $A \perp X$ ("A remainder X") is the set of inclusion-maximal subsets of A that do not imply any element of X. In other words, a set B is an element of $A \perp X$ if and only if B is a subset of A that does not imply any element of X, and there is no set B' not implying any element of X such that $B \subset B' \subseteq A$. For any sentence p, we define $A \perp p = A \perp \{p\}$.

In one of the proofs we will make use of the *upper bound property*, namely: If $X \subseteq A$ and $X \nvdash B$, then there is some X' such that $X \subseteq X' \in A \perp B$. As was observed by Alchourrón and Makinson (1981, p.129), the upper bound property follows from compactness and Zorn's lemma.

Expansion, denoted +, is the operation such that $K + p = Cn(K \cup \{p\})$. *Full meet contraction, denoted ~, is the operation such that if* $K \perp X$ *is non-empty, then* $K \sim X = \bigcap(K \perp X)$ *and otherwise* $K \sim X = K$. *Furthermore,* $K \sim p = K \sim \{p\}$.

3. The seven virtues of full meet contraction

The following seven properties of full meet contraction all contribute to making it a useful building-block in the construction of contraction operators.

The first virtue: Full meet contraction is a purely logical operation

Most contraction operators include some non-logical selection mechanism, such as a selection function (Alchourrón, Gärdenfors, and Makinson 1985), a hierarchy (Alchourrón and Makinson 1985, 1986), an incision function (Hansson 1994) or an entrenchment ordering (Gärdenfors 1988, Gärdenfors and Makinson 1988). Full meet contraction differs from the other contraction operators in being a purely logical operator that contains no mechanism for non-logical selection. This, of course, makes it unrealistic as a representation of actual patterns of contraction. However, in the construction of composite contraction operators, this is a valuable property since it makes sense to separate the logical from the non-logical components of belief contraction so that they can be characterized separately. Full meet contraction appears to be the only purely logical contraction operator that is available for this purpose.

The second virtue: Full meet contraction retains finite-based-ness of the belief set

It is a reasonable criterion of cognitive realism that the belief set is finite-based. Furthermore, if K is finite-based, then we should expect the contraction outcome $K \div p$ to be finite-based as well. This, however, does not hold for partial meet contraction. The reason for this is that if the language is infinite (and even if K is finite-based), the process of forming remainders gives rise to infinity on two levels: the set of remainders is infinite and none of the individual remainders is finite-based.

OBSERVATION 2. (Hanssson 2006a) Let the language consist of infinitely many logically independent atoms and their truth-functional combinations. Let K be a belief set that contains some non-tautology, and let $p \in K \setminus Cn(\varnothing)$. Then:

(1) $K \perp p$ is infinite, and
(2) if $X \in K \perp p$, then X is not finite-based.

COROLLARY. Let the language consist of infinitely many logically independent atoms and their truth-functional combinations. Let K be a belief set that contains some non-tautology, let $\div$ be a maxichoice operation on K, and let $p \in K \setminus Cn(\varnothing)$. Then $K \div p$ is not finite-based.

PROOF. See Hansson 2006a.

Observation 2 is closely connected with a cognitive problem for partial meet contraction: Even in intuitively trivial cases, such as choosing in the contraction by *p&q* whether to retain *p*, *q*, or neither of them, the selection process takes the form of selecting among an infinite array of infinite entities (the remainders). This is clearly not a cognitively realistic representation of the selection problem.

In contrast to partial meet contraction, the application of full meet contraction never gives rise to loss of finite-basedness (and, consequently, neither does specified meet contraction).

OBSERVATION 3. (Hansson, 2006a) If K is finite-based, then so is $K \sim p$ for all p.

PROOF. Due to Observation 1, $K \sim p = K \cap Cn(\{\neg p\})$, and since K is finite-based we then have $K \sim p = Cn(\{\&K \vee \neg p\})$.

The third virtue: Full meet contraction is the inclusion-maximal contraction that removes all sentences that can contribute to implying the input sentence.

A contraction $K \div p$ involves the removal not only of p but also of other sentences in K that, singly or in combination, imply p. It would for instance be inadequate to retain a sentence p & q when contracting by p, or to retain both $q \rightarrow p$ and $q \vee p$. We can of course retain one of $q \rightarrow p$ and $q \vee p$ if we remove the other, but that involves a choice based on non-logical criteria. Full meet contraction has a simple and intuitive characterization as the contraction that removesall sentences that can contribute to making a set imply p, but removes nothing else. In other words, $K \div p$ consists of all those sentences in K that can be added to any set not implying p without making the resulting set imply p.

OBSERVATION 4. (Hansson, 2006b) $x \in K \sim p$ if and only if $x \in K$ and for all sets Z of sentences, if $Z \cup \{x\} \vdash p$ then $Z \vdash p$.

COROLLARY. For all q, if $K \sim p + q \vdash p$, then $q \vdash p$.

PROOF.
$x \in K$ and for all Z, if $Z \cup \{x\} \vdash p$ then $Z \vdash p$
iff: $x \in K$ and for all z, if $z \vdash x \rightarrow p$ then $z \vdash p$
iff: $x \in K$ and $\vdash (x \rightarrow p) \rightarrow p$

iff: $x \in K$ and $\vdash x \vee p$
iff: $x \in K$ and $\vdash (p \rightarrow \& K\ ) \rightarrow x$ (Since $K \vdash x$)
iff: $x \in K$ and $x \in K \sim p$ (Observation 1)
iff: $x \in K \sim p$

The fourth virtue: Full meet contraction is the inclusion-maximal contraction that removes all non-tautologous consequences of the input sentence.

Full meet contraction also has an intuitive interpretation in terms of sentences implied by *p* (rather than sentences implying *p*). Clearly, when removing a sentence *p* we cannot remove all its logical consequences, since it implies all tautologies. We can, however, remove all its non-tautological consequences. As the following observation shows, full meet contraction does so, and it is also the inclusion-maximal operator with this property.

DEFINITION 1. $\div$ satisfies complete eradication if and only if:
For all $p \in K\ $, if $p \vdash q$ and q then $q \notin K \div p$.

OBSERVATION 5. (1) (Hansson 2006b) Full meet contraction satisfies complete eradication.

(2) Let $\div$ be an operation on $K\ $ that satisfies inclusion (i.e. $K \div p \subseteq K$ for all p). Then $\div$ satisfies complete eradication if and only it holds for all $p \in K\ $ that $K \div p \subseteq \sim p$.

PROOF. Part 1: Complete eradication holds vacuously if $p \in Cn(\varnothing)$. In the principal case, when $p \in K \setminus Cn(\varnothing)$, suppose to the contrary that $p \vdash q$, $\nvdash q$ and $q \in K \sim p$. We then have:
$q \in Cn(\{p \rightarrow \& K\ \})$ (Observation 1)
$\vdash (p \rightarrow \& K\ ) \rightarrow q$
$\vdash q$ (Since $p \vdash q$))
Contradiction.

Part 2: For one direction, it follows from Part 1 that if $K \div p \subseteq K \sim p$ for all $p \in K$, then $\div$ satisfies complete eradication.

For the other direction, suppose to the contrary that $\div$ satisfies complete eradication and that there is some s such that $s \in (K \div p) \setminus (K \sim p)$. Since $K \div p \subseteq K$, we have $s \in K$ and due to the recovery property of $\sim$ (Alchourrón and Makinson 1985, Hansson 1999, p.71) $p \rightarrow s \in K\ \sim p$. It follows from this and $s \notin K\ \sim p$ that $\nvdash p \vee s$. It follows from $s \in K \div p$ that $p \vee s \in K \div p$, which contradicts complete eradication.

Since full meet contraction by a sentence removes all its non-tautological consequences, it is not surprising that full meet contraction by *p* coincides with full meet contraction by the set consisting of all the non-tautological consequences of *p*.

OBSERVATION 6. (Hansson, 2006b) If the logic is compact and Zorn's lemma is satisfied, then: If $p \in K$, then $K \sim p = K \sim (Cn(\{p\}) \setminus Cn(\emptyset))$.

PROOF. See Hansson, 2006b.

The fifth virtue: Almost all contractions can be reconstructed as full meet contraction.

A surprisingly wide range of contraction operators can be constructed as composite operations using full meet contraction as a building-block, according to the recipe already referred to, namely $K \div p = K \sim f(p)$ (specified meet contraction). The following observation provides the exact conditions for when this can be done.

DEFINITION 2. An operator ÷ satisfies eradicative reconstruction if and only if for all p there is some p' such that $K \div p = K \sim p'$.

OBSERVATION 7. (Hansson, 2006b) An operation ÷ on a finite-based belief set K satisfies eradicative reconstruction if and only if it satisfies:

$K \div p \subseteq K$ (inclusion)

$K \div p = Cn(K \div p)$ (closure)

$K \div p$ is finite-based (finite-based outcomes)

PROOF. For one direction, let $K \div p = K \sim p'$. Inclusion and closure follow from the properties of full meet contraction. It follows from Observation 3 that finite-based outcomes are satisfied.

For the other direction, let ÷ satisfy the three postulates. Due to finite-based outcomes, $\&(K \div p)$ is well-defined. We can therefore define p' so that:

If $K \div p \subset K$, then: $p' = \&(K \div p) \rightarrow \&K$.

Otherwise: $p' \notin K \setminus Cn(\emptyset)$.

In order to verify the construction we need to show that the identity $K \div p = K \sim p$ holds. Due to inclusion, either $K \div p = K$ or $K \div p \subset K$.

In the former case, it follows directly from our definition of p' and the properties of full meet contraction that K~ p' = K. In the latter case we have:

$K \div p = Cn(\{\&(K \div p)\})$ (closure, finite-based outcomes)
$= Cn(\{\& K \vee \neg(\&(K \div p) \rightarrow \& K)\})$
(Since $\vdash \&K \rightarrow \&(K \div p)$ that follows from inclusion)
$= Cn(\{\&K\}) \cap Cn(\{\neg(\&(K \div p) \rightarrow \&K)\})$
$= K \cap Cn(\{\neg(\&(K \div p) \rightarrow \&K)\})$
$= K \sim (\&(K \div p) \rightarrow \&K)$ (Observation 1)
$= K \sim p'$.

Inclusion is a highly plausible requirement for an operator of contraction. An operator that adds something new to the belief set is not a contraction in the proper sense of the word. Both closure and finite-basedness require that the contraction outcome should have the same type of belief representation as the initial belief set. A contraction of a finite-based, logically closed set should result in a new finite-based, logically closed set, not in a belief set that has no finite base or in a set that is not logically closed.

It can therefore be concluded from Observation 7 that all plausible contraction operators can be reconstructed as full meet contraction via a sentential selector. Another way to express this is that specified meet contraction is sufficiently general to cover all plausible contraction operations.

The sixth virtue: Full meet contraction allows for recovery of the input sentence.

If two sentences are logically equivalent, then contractions by them give rise to the same contraction outcome. On the other hand, contraction by non-equivalent sentences in K always yield non-identical outcomes.

OBSERVATION 8. Let $p, q \in K$. Then $K\sim p = K\sim q$ if and only if $\vdash p \leftrightarrow q$.

PROOF. For one direction, let $K\sim p = K \sim q$. It follows from Observation 1 that $\vdash (p \rightarrow \&K) \leftrightarrow (q \rightarrow \& K)$. From this, $\vdash \&K \rightarrow p$, and $\vdash \&K \rightarrow q$ it follows that $\vdash p \leftrightarrow q$.

For the other direction, let $\vdash p \leftrightarrow q$. Then $K\sim p = K \sim q$ follows directly from the definition of full meet contraction.

Thus, no distinctions between elements of K are lost in full meet contraction. Therefore, it should be possible to regain the contracted sentence p whenever K and $K \sim p$ are known. The following observation shows how this can be done.

OBSERVATION 9. Let $p \in K \setminus Cn(\emptyset)$. Then $\vdash p \leftrightarrow (\&(K \sim p) \rightarrow \&K)$

PROOF. Due to Observation 1, $\&(K \sim p) \rightarrow \&K$ is logically equivalent to $(p \rightarrow \&K) \rightarrow \&K$. Since $\&K \vdash p$, $(p \rightarrow \&K) \rightarrow \&K$ is equivalent to p.

Observations 1 and 9 give rise to a nice symmetry between a sentence and its corresponding contraction outcome:

$\vdash \&(K \sim p) \leftrightarrow (p \rightarrow \&K)$
$\vdash p \leftrightarrow (\&(K \sim p) \rightarrow \&K)$

The following observation further confirms this symmetry:

OBSERVATION 10. Let $p,q \in K \setminus Cn(\emptyset)$. Then:
(1) $\vdash q \leftrightarrow \&(K \sim p)$ if and only if $\vdash p \leftrightarrow \&(K \sim q)$
(2) $q \in K \sim p$ if and only if $p \in K \sim q$
(3) $\vdash p \leftrightarrow K \sim \&(K \sim p)$

PROOF. Part 1:
$\vdash q \leftrightarrow \&(K \sim p)$
iff $\vdash q \leftrightarrow (p \rightarrow \&K)$ (Observation 1)
iff $\vdash p \leftrightarrow (q \rightarrow \&K)$ (Since $\&K \vdash p$ and $\&K \vdash q$)
$\vdash p \leftrightarrow \&(K \sim q)$ (Observation 1)
Part 2:
$Q \in K \sim p$
iff $\vdash (p \rightarrow \&K) \rightarrow q$ (Observation 1)
iff $\vdash (q \rightarrow \&K) \rightarrow p$ (Since $\&K \vdash p$ and $\&K \vdash q$)
iff $p \in K \sim q$ (Observation 1)
Part 3:
$Cn(\{p\}) = Cn(\{(p \rightarrow \&K) \rightarrow \&K\})$ (Since $\&K \vdash p$)
$= Cn(\{\&(K \sim p) \rightarrow \&K\})$ (Observation 1)
$= K \sim \&(K \sim p)$ (Observation 1)

The seventh virtue: Full meet contraction provides a unified account of multiple and singleton contraction.

Multiple full meet contraction, i.e. the simultaneous full meet contraction by several sentences (Fuhrmann and Hansson, 1994), can be reduced to full meet contraction by a single sentence. The following definition is needed for the reduction:

DEFINITION 3. For any set B of sentences:
nimp (B) = $\{x \in B \mid (\forall y \in B)$ (If $x \vdash y$ then $y \vdash x$)$\}$

In other words, a sentence is in nimp (B) (the "non-impliers" in B) if and only if it is an element of B that implies no other element of B than itself (or elements to which it is equivalent).

OBSERVATION 11. If compactness and Zorn's lemma hold, then:
If B is finite, then $K \sim B = K \sim \&$(nimp $(K \cap B)$)

PROOF: Step 1: The first step is to note that $K \perp B = K \perp (B \cap K)$. A subset of K cannot imply any sentence that is not in K, therefore it implies some element of B if and only if it implies some element of $B \cap K$.

Step 2: The next step is to note that for any subset B' of K (such as $B' = B \cap K$), $K \perp B' = K \perp$ nimp (B'). This follows since any set X implies some element of B' if and only if it implies some element of nimp (B').

We can conclude from steps 1 and 2 that $K \perp B = K \perp$ nimp $(B \cap K)$, thus $K \sim B = K \sim$ nimp $(K \cap B)$.

Step 3: Finally we are going to show that $K \sim$ nimp $(B \cap K) = K \sim \&$(nimp $(K \cap B)$).

Let D = nimp $(B \cap K)$ and note that D is a subset of K such that no element of D implies another element of D unless they are equivalent.

In order to show that $K \sim \&D \subseteq K \sim D$, let $z \in K \sim \&D$. It follows from Observation 1 that $\neg\& D \vdash z$, thus it holds for all $d \in D$ that $\vdash d \vee z$.

Now suppose that $z \notin K \sim D$. Then since $z \in K$ there is some X such that $z \notin X \in K \perp D$. Thus there must be some $d \in D$ such that $X \cup \{z\} \vdash d$. Thus $z \rightarrow d \in X$. Since $X \nvdash d$, we have $(z \rightarrow d) \rightarrow d \notin X$, i.e. $d \vee z \notin X$, contrary to $\vdash d \vee z$. We can conclude from this contradiction that $z \in K \sim D$.

For the other direction, i.e. in order to show that $K \sim D \subseteq K \sim \&D$, let $z \notin K \sim \&D$. Then there is some X such that $z \notin X \in K \perp \&D$. It follows that X $\&D$, and thus there is some $d \in D$ such that $X \nvdash d$. However, it also follows that $X \cup \{z\} \vdash \&D$, and thus $X \vdash z \rightarrow d$. It follows from $X \nvdash d$ and $X \vdash z \rightarrow d$ that $z \rightarrow d \nvdash d$. Suppose that there is some other $d' \in D$ such that $z \rightarrow d \vdash d'$. It would then follow that $d \vdash d'$, contrary to the structure of D. We can conclude that $z \rightarrow d$ implies no element of D. It follows from the upper bound property (see Section 2) that there is some Y such that $z \rightarrow d \in Y \in K \perp D$. Clearly, $z \notin Y$. It follows that $z \notin K \sim D$.

It follows from Observation 7 that the outcome of any series of contractions by single sentences coincides with the outcome of contraction by one single sentence, i.e. for any series $p_1, p_2, \ldots p_n$ of sentences there is a sentence q such that $K \sim p_1 \sim p_2 \sim\ldots\sim p_n = K \sim q$. However, the outcome of iterated full meet contraction is order-dependent, and therefore it cannot be described in the same unified fashion as that of multiple full meet contraction.

OBSERVATION 12. Let $p_1, p_2 \in K$. Then:
(1) If $\vdash p_1 \vee p_2$, then $K\sim p_1 \sim p_2 = K\sim p_2 \sim p_1 = K \sim (p_1 \& p_2)$.
(2) If $\nvdash p_1 \vee p_2$ then $K \sim p_1 \sim p_2 = K \sim p_1$.

PROOF. Part 1: Let $\vdash p_1 \vee p_2$. We then have:
$K \sim p_1 \sim p_2 = (Cn(\{p_1 \rightarrow \&K\})) \sim p_2$ (Observation 1)
$= Cn(\{p_2 \rightarrow (p_1 \rightarrow \&K)\})$
(Observation 1. $p_1 \rightarrow \&K \vdash p_2$ follows from $\vdash p_1 \vee p_2$ and $\&K \vdash p_2$.)
$= Cn(\{p_1 \& p_2 \rightarrow \&K\})$

$= K\sim (p_1 \& p_2)$ (Observation 1)

The proof that $K \sim p_2 \sim p_1 = K \sim (p_1 \& p_2)$ is similar.

Part 2: Since $\&K \vdash p_2$, it follows from $\nvdash p_1 \vee p_2$ that $\nvdash (p_1 \rightarrow \&K)$. Thus, according to Observation 1, we have $p_2 \notin K \sim p_1$, thus $K \sim p_1 \sim p_2 = K \sim p_1$.

4. Conclusions

We have shown that full meet contraction is a purely logical operator that allows us to separate out the logical from the non-logical components of a composite contraction operator (1). Almost all contractions can be reconstructed as full meet contraction; the only exceptions are contractions with the implausible property that take us from a finite-based belief set to one that is not finite-based (2 and 5). The use of full meet contraction in composite operations is facilitated by the fact that it has highly plausible intuitive interpretations, both in terms of sentences implying and sentences implied by the sentence to be contracted (2 and 5).

Furthermore, distinctions between input sentences are preserved in full meet contraction and the input sentence can be recovered from the contraction output (6). Finally, multiple full meet contraction can be reconstructed as contraction by a single sentence (7).

In conclusion, full meet contraction has an interesting set of properties that make it eminently well suited as a component in composite contraction operators. There is no better proof of the richness of the early work in belief revision by Carlos Alchourrón and his co-workers than its usefulness, decades later, as a direct inspiration for new developments in the area.

5. References

Alchourrón, Carlos, Peter Gärdenfors, and David Makinson (1985) "On the Logic of Theory Change: Partial Meet Contraction and Revision Functions", *Journal of Symbolic Logic* 50: 510-530.

Alchourrón, Carlos and David Makinson (1981) "Hierarchies of Regulation and Their Logic", pp. 125-148 in Risto Hilpinen (ed.) *New Studies in Deontic Logic,* D. Reider Publishing Company.

——— (1982) "On the Logic of Theory Change: Contraction Functions and Their Associated Revision Functions", *Theoria* 48: 14-37.

——— (1985) "On the Logic of Theory Change: Safe Contraction", *Studia Logica*, 44: 405-422.

——— (1986) "Maps Between Some Different Kinds of Contraction Function: The Finite Case", *Studia Logica,* 45: 187-198.

Fuhrmann, André and Sven Ove Hansson (1994) "A Survey of Multiple Contraction", *Journal of Logic, Language and Information* 3:39-76.

Gärdenfors, Peter (1988) *Knowledge in Flux. Modeling the Dynamics of Epistemic States*, The MIT Press, Cambridge, Massachusetts.

Gärdenfors, Peter and David Makinson (1988) "Revisions of Knowledge Systems Using Epistemic Entrenchment" In Moshe Y. Verdi (ed.). *Proceedings of the Second Conference on Theoretical Aspects of Reasoning about Knowledge,* pp. 83-95, Morgan Kaufmann, Los Altos.

Hansson, Sven Ove (1994) "Kernel Contraction", *Journal of Symbolic Logic* 59:845-859.

——— (1999) *A Textbook of Belief Dynamics. Theory Change and Database Updating,* Kluwer.

——— (2006a) "Specified Meet Contraction", submitted manuscript.

——— (2006b) "Contraction Based on Sentential Selection", submitted manuscript.